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**New Form of the Region-to-Factor Message**

1. **New Form of the Region-to-Factor Message**

Previously, we proposed the following form of the region-to-factor message,



However, this gives rise to recursion relations with terms difficult to interpret. We need to explore more convenient forms of the region-to-factor messages.

Some insights can be obtained from the Edwards-Anderson model [1]. In that model, a factor-to-node message is a single-spin message contributing to the magnetic field of a node, and a region-to-factor message consists of two single-spin messages passed to the two nodes belonging to the factor, and a two-spin message passed to the factor. The single-spin messages contribute to the magnetic fields of the nodes, and the two-spin message contributes to the coupling between the nodes. The region-to-factor message then corresponds to a spin chain with modified coupling strengths, with each spin subject to magnetic fields due to different contributions.

Extending to the network flow model, a factor-to-node message corresponds to a current supply through a potential and a conductance. For the region-to-factor message, it is natural to propose that there are two single-potential messages that supply current through a potential and a conductance to the two nodes belonging to the factor, and a double-potential message that contributes to the conductance of the link between the nodes. The region-to-factor message then corresponds to a chain of nodes, with modified conductance and each node fed by a few current supplies. Hence we propose



Note that this region-to-factor message consists of 5 parameters. This is the same number of parameters as in the previous form. Hence there is a one-to-one correspondence between them.

1. **Factor-to-Node Messages**

Consider generalized belief-propagation. A factor collects factor-to-node messages from factors connected from nodes 2, 3 and 4 neighboring node *j* and region-to-factor messages from regions *p* and *q* feeding the factor. The factor then sends a message from node *j* to node *i*.



***i***

**2**

**4**

***j***

***p***

***q***

**3**

The generalized belief-propagation equation becomes



*g* is inserted for checking. It will be set to *g* = 1 at the end. Assume that the messages have the forms





Minimization equation:



Introduce the conductances and the potential

, , 

Then the equation is simplified to





The factor-to-node message becomes







Quadratic term in *Fj*(*μi*)

Linear term in *Fj*(*μi*) 

Further introduce the conductance and potential

, 

The recursion relation becomes

, , , ,



 and



Note that  satisfies the equation .

Hence *gij* and  can be interpreted from the following equivalent circuit:

***i***

***Vj***

**Λ*j***

***Vp***

***Gj***

***Gij***

***Gp***



***μj***

*gij* is the conductance of *Gij* and *Gj* in series, connected with *Gp* in parallel.

 is the potential of *i* satisfying Kirchhoff’s law

, 

Eliminating *μj*, we obtain the same result as the recursion relations.

1. **Region-to-Factor Messages**

Consider a region *p* consisting of factors *a*, *b*, *c* and *d*. It collects 4 factor-to-node messages from factors connected from nodes 3 and 4, 7 and 8 to nodes 5 and 6 respectively, and 3 region-to-factor messages feeding respectively nodes 2 and 5, 5 and 6, 6 and 1. The region then sends a message to factor 1 and 2.

**7**

**8**

**4**

**61**

**25**

**56**

**12**



**1**

**2**

**5**

**3**

**6**

Then the generalized belief-propagation equation becomes







Minimization equation:









Introduce the conductances and the potentials















In matrix form,



Solution:



The region-to-factor message becomes, after neglecting constant terms,









Coefficient of 

Coefficient of 

Coefficient of 

Coefficient of  



Coefficient of  



The recursion relation becomes, immediately after updating 

, ,

, ,

, , ,

,











Note that  and  satisfy the following equations









The equivalent circuit is given by

**Λ5**

***G*52**



***μ*5**

***G*56**

***G*5**

***G*61**

***G*6**

***g*252**

***g*611**

***−h*12**

***−g*16**

***−g*25**

**Λ6**



***V*5**

***V*6**





***μ*6**





[1] E. Dominguez, A. Lage-Castellanos, R. Mulet, F. Ricci-Tersenghi, T. Rizzo, “Characterizing and Improving Generalized Belief Propagation Algorithms on the 2D Edwards-Anderson Model”, J. Stat. Mech. P12007 (2011). arXiv:1110.1259